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A BOUND FOR THE DIAMETER OF DISTANCE-REGULAR GRAPHS

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We prove that every distance-regular graph on n vertices has diameter at most $5\log n$. This essentially settles a problem of Brouwer, Cohen and Neumaier.

0. Introduction

Let G be a connected graph on n vertices. For two vertices x,y of G we denote by $d(x,y)$ the distance between x and y in G. We write $\Gamma_i(x)$ for the set of vertices y with $d(x,y)=i$. Instead of $\Gamma_1(x)$ we write $\Gamma(x)$.

For two vertices x,y at distance j we write $c_j(x,y)=|\Gamma_{j-1}(x)\cap\Gamma(y)|$, $a_j(x,y)=|\Gamma_j(x)\cap\Gamma(y)|$ and $b_j(x,y)=|\Gamma_{j+1}(x)\cap\Gamma(y)|$. G is said to be *distance-regular* if the numbers $c_j(x,y)$, $a_j(x,y)$ and $b_j(x,y)$ depend only on the distance j of x and y. In this case we denote these numbers simply by c_j, a_j and b_j.

Let $d=d(G)$ be the diameter of G and $k=b_0$ the valency of G, then the array

$$\begin{Bmatrix} 0 & 1 & \dots & c_i & \dots & c_{d-1} & c_d \\ 0 & a_1 & \dots & a_i & \dots & a_{d-1} & a_d \\ k & b_1 & \dots & b_i & \dots & b_{d-1} & 0 \end{Bmatrix}$$

is the *intersection array* of G. We will always assume that $k>2$.

Various bounds for the diameter of distance-regular graphs are known (see [3] Chapter 5). For example Ivanov's theory [4], yields a bound on the diameter of distance-regular graphs with fixed valency and girth.

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As a by-product of this theory Brouwer, Cohen and Neumaier prove that $n > d^2$ for $d \geq 20$ and $k > 2$ and ask for a bound for d which is logarithmic in n (see [3] p. 189 for a more precise formulation). Note that for d-cubes we have $n = 2^d$ hence in this case the diameter is exactly $\log n$ (throughout this paper log denotes logarithm to the base 2).

Our main result is the following.

Theorem. *The diameter of a distance-regular graph G on n vertices is at most $5\log n$.*

The Theorem can be used to give a lower bound for the expansion of a distance-regular graph G and to obtain upper bounds for the order of its automorphism group (along the lines of [2] and [1]). These applications will be discussed elsewhere.

1. The Proof

Let us list a few well-known properties of intersection arrays (cf. [3]).

Proposition 1.1.

(i) $k = b_0 > b_1 \geq b_2 \geq \ldots \geq b_{d-1} \geq 1$;

(ii) $1 = c_1 \leq c_2 \leq \ldots \leq c_d \leq k$;

(iii) $c_i \leq b_j$ *if* $i + j \leq d$.

Recently Koolen [5] obtained a useful new general restriction.

Lemma 1.2. *Let $e \geq 2$ be an integer. Let G be a distance-regular graph such that $c_{e-1} < c_e$. Then*

$$c_i + c_{e-i} \leq c_e \quad \textit{for all} \quad i \qquad 1 \leq i \leq e-1.$$

Our main observation is that a "weak dual" of Koolen's lemma holds.

Lemma 1.3. *Let e and i be integers such that $e > i \geq 1$. Let G be a distance-regular graph such that $b_{d-e} > b_{d-e+1}$. Then we have the following*

(i) *If x, u, u', x' are vertices of G satisfying $d(x,u) = i$, $d(u,u') = d-e$, $d(u',x') = e-i$ and $d(x,x') = d$ then there exist a bijective map $\Phi : \Gamma(u) \cap \Gamma_{d-e+1}(u') \to \Gamma_{d-e+1}(u) \cap \Gamma(u')$ such that $d(v, \Phi(v)) \leq d-e+1$;*

(ii) $b_{d-e} \geq c_{e-i} + c_i$.

Proof. Set $S = \Gamma(u) \cap \Gamma_{d-e+1}(u')$ and $S' = \Gamma_{d-e+1}(u) \cap \Gamma(u')$. Define the set P_s by $P_s = \{s' \in S' | d(s,s') \leq d-e+1\}$ for $s \in S$. We define $P_{s'}$ analogously for $s' \in S'$. Setting $B(s,u') = \Gamma_{d-e+2}(s) \cap \Gamma(u')$ we have $|B(s,u')| = b_{d-e+1}$.

Claim.

$$B(s,u') = S' \setminus P_s.$$

If $s' \in S' \setminus P_s$ then $d(s,s') = d-e+2$ and $s' \in \Gamma(u')$ hence $s' \in B(s,u')$ as required.

If $b \in B(s,u')$ then $b \in \Gamma(u')$, $b \notin P_s$ hence we have to show that $b \in \Gamma_{d-e+1}(u)$. But $d(s,b) = d-e+2$ implies $d(u,b) \geq d-e+1$ and $d(u',b) = 1$ implies $d(u,b) \leq d-e+1$.

This proves the claim.

By the above claim (and its analogue for s') we have $|P_s| = |P_{s'}| = b_{d-e} - b_{d-e+1}$. Let Δ be a bipartite graph with vertex set $S \cup S'$ such that (s,s') is an edge of B exactly if $d(s,s') \leq d-e+1$. Then Δ is a regular bipartite graph and thus has a perfect matching. This proves (i).

Setting $C(x,u) = \Gamma_{i-1}(x) \cap \Gamma(u)$ and $C(x',u') = \Gamma_{e-i-1}(x') \cap \Gamma(u')$ we have $|C(x,u)| = c_i$ and $|C(x',u')| = c_{e-i}$.

We claim that $C(x,u) \subset S$. If $c \in C(x,u)$ then $c \in \Gamma(u)$ hence it suffices to show that $d(c,u') = d-e+1$. But this follows from the inequalities $d(c,u') \leq 1 + d(u,u') = d-e+1$ and $(i-1) + d(c,u') + e - i \geq d$.

Similarly we have $C(x',u) \subset S'$.

Now $C(x',u')$ and $\Phi(C(x,u))$ are disjoint for otherwise we would have

$$d = d(x,x') \leq (i-1) + (d-e+1) + (e-i-1)$$

which is impossible. Hence $c_i + c_{e-i} \leq |S'| = b_{d-e}$ as required. ∎

Note that for antipodal graphs 1.2 and 1.3 (ii) are essentially equivalent (see [3] 4.2.2).

We will also use a crucial result of Ivanov [4] (cf. [3] 5.9.6).

Lemma 1.4. *Let G be a distance-regular graph and $i > 1$. Suppose the intersection array of G satisfies $(c_{i-1}, a_{i-1}, b_{i-1}) \neq (c_i, a_i, b_i) = \ldots = (c_{i+j}, a_{i+j}, b_{i+j})$ then*

(i) *$j+1 \leq i$;*

(ii) *if in addition $j+2 \leq i$ then $b_j \geq 2c_{j+1}$.*

Note that the conclusions of the lemma are valid if we assume $(c_1, a_1, b_1) \neq (c_i, a_i, b_i)$ instead of $(c_{i-1}, a_{i-1}, b_{i-1}) \neq (c_i, a_i, b_i)$.

In a distance-regular graph the numbers $|\Gamma_i(x)|$ do not depend on the choice of x hence we can denote $|\Gamma_i(x)|$ by k_i. We have $\frac{k_{i+1}}{k_i} = \frac{b_i}{c_{i+1}}$ and using Proposition 1.1 this immediately implies that $\frac{k_{i+1}}{k_i} \geq \frac{k_{i+2}}{k_{i+1}}$ for all i.

Proposition 1.5. *Let G be a distance-regular graph with valency $k>2$ and diameter $d>2$ and let ℓ denote the smallest index for which $\frac{k_{\ell+1}}{k_\ell}<2$. Then*

(i) $\ell \le \log k_\ell$;

(ii) *if the intersection array of G satisfies*

$$(c_i, a_i, b_i) = \ldots = (c_{i+j+1}, a_{i+j+1}, b_{i+j+1})$$

for some $j \ge 0$, then $j+1 \le \ell$.

Proof. As noted above the ratios $\frac{k_{i+1}}{k_i}$ are monotone decreasing and therefore we have $k \cdot 2^{\ell-1} \le k_\ell$. This proves (i).

The hypothesis on the intersection array implies that $\frac{k_{i+1}}{k_i} = \frac{k_{i+2}}{k_{i+1}} = \ldots = \frac{k_{i+j+1}}{k_{i+j}}$. By the choice of ℓ we have $\frac{k_\ell}{k_{\ell-1}} > \frac{k_{\ell+1}}{k_\ell}$ and if $i<\ell$ this implies $i+j+1 \le \ell$.

If $i \ge \ell$ then we have $\frac{b_i}{c_{i+1}} \le \frac{b_\ell}{c_{\ell+1}} = \frac{k_{\ell+1}}{k_\ell} < 2$. If $(c_1, a_1, b_1) = (c_i, a_i, b_i)$ then $c_{i+1} = c_1 = 1$ implies that $b_i < 2$ and hence $b_1 = b_i = 1$. If $b_1 = c_1 = 1$ then G is a polygon [3, p. 188] which we have excluded. Using Lemma 1.4 (i) we obtain that $j+2 \le i$ and using 1.4 (ii) that $\frac{k_{j+1}}{k_j} = \frac{b_j}{c_{j+1}} \ge 2$.

Therefore in both cases we have $j<\ell$ as required. ∎

The proof of the Theorem

Let ℓ be as in Proposition 1.5. Consider the numbers c_x, b_x such that $2\ell+2 \le x \le 3\ell+2$. By Proposition 1.5 we either have $c_{x-1} < c_x$ or $b_{x-1} > b_x$ for some x.

In the first case using Koolen's lemma we obtain that $c_x \ge 2c_{\ell+1} > b_\ell$. By 1.1 (iii) this gives us $d < x+\ell \le 4\ell+2$.

In the second case if $d \ge 5\ell+3 \ge (x-1)+2(\ell+1)$ then using Lemma 1.3 we obtain that $b_\ell \ge b_{x-1} \ge 2c_{\ell+1}$ a contradiction. Therefore in both cases we have $d \le 5\ell+2$.

If $k_\ell \le \frac{n}{2}$ then using 1.5 (i) we obtain that $d \le 5\log n$. If $k_\ell > \frac{n}{2}$ then $\frac{b_\ell}{c_{\ell+1}} = \frac{k_{\ell+1}}{k_\ell} < 1$ and therefore by 1.1 (iii) we have $d \le 2\ell \le 2\log n$. The proof is complete. ∎

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